The Lonely Runner Conjecture

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Abstract

Suppose there are n runners in a circle, initially at the same position. Assume the circle has unit length, and all the runners start with constant, but distinct speeds. A runner is said to be *lonely* at a time if the person is away from all others at least by a distance of $1/n$ unit. The lonely runner conjecture states that every runner will be lonely at some point in time. In this note, we will discuss this unsolved problem and present the known results so far.

Introduction

Imagine a circular track, with unit circumference for simplicity. Let there be two runners at the starting point. This is not a usual running game in which the first one who reaches the finishing point is the winner. Instead, we are considering the case where the runners have a constant speed throughout; also, the speed of each runner is different from the other. Moreover, the runners continue to run even after finishing a lap – they keep on running through the circular track without stopping. This means that the runner with the greater speed among them is ahead while the other runner is behind. At some point in time, the runner who is ahead will cross the other runner since they are not stopping. Hence, there will be a time t_0 at which the faster runner is ahead of the slower runner by a distance of ½ unit measured along the circumference of the track. This means, they are away from each other by ½ unit, or one is *lonely* in the sense that the other person is away by a distance of ½ unit.

Now we can generalize this situation to the case of n runners. For this, we need to modify the definition of *loneliness* accordingly. If there are n runners, we say a runner is lonely if the person is away from all others at least by a distance of $1/n$ unit. We are now ready to state the conjecture:

Conjecture (Lonely runner): *If there are* n *runners on a circular track of unit length, running with constant but distinct speeds, then every runner will be lonely at some point in time.*

In this article, we look at the history of the problem and its connection to various fields of mathematics. We also present the known results so far with sketches of proofs in a few cases.

A Reformulation of the Conjecture

The above formulation of the conjecture was made by Bienia et. al. [3]. Though, it was originally stated by J. M. Wills in 1967[9] as a Diophantine approximation problem. We have a reformulation of the conjecture (in its original form) as follows:

Conjecture: *Let* $a_1, a_2, \ldots, a_{n-1}$ *be positive real's. Then we can find another real number* $t>0$ *such that*

$$\{a_i t\} \in \left(\frac{1}{n}, \frac{(n-1)}{n}\right) \text{ for } i=1\ldots,n-1,$$

where $\{x\}$ *is the fractional part of x, i.e.,* $x = x - \lfloor x \rfloor$.

We would like to point out that although the problem is easy to understand without any serious knowledge of mathematics, the solution is not easy. We will start with the simple cases and point out the proof attempts in the remaining cases which require careful analysis.

The Cases of 1 & 2 Runners

The conjecture is obviously true for *n*=1 because the sole runner is always lonely at every time. Now, in the case of two runners, both are at different speeds. At some point of time, they will be diametrically opposite to each other. So, the difference between their distance will be ½, they are lonely.

The Case of 3 Runners

We then evaluate the situation with three runners. A stationary runner experiences loneliness at some point, and that is what we need to demonstrate. The other two runners are A and B, and A is the fastest of the two. B is located 1/3 of the way from the stationary position in the first stage. Only when A is no slower than B than C can this concept be applied. Now, if A is moving at a speed more than twice that of B, we can observe that B is currently between the 1/3 and 2/3 points of A's speed. A travel more than 2/3 of the distance in this interval; therefore, at some point, he will also be further than 1/3 from the beginning [10].

Remaining Cases up to 7 Runners

For 4 runners, several proofs are available in the context of the Diophantine approximation problem, see [2,5]. The first proof for the case of 5 runners was given by Cusick and Pomerance, in connection with view-obstruction problems. This was computer-assisted proof. Another independent proof was given in [3] later. The cases of 6 and 7 runners were done in [4] and [1] respectively.

The conjecture remains open for the remaining cases.

Related Results

There are several reformulations to the conjecture (one such was given after the introduction). The problem remains the same if we reduce the speed of one runner from all others - taking the speed of that runner to be zero and look at the time at which this runner becomes lonely. So, if there are *n* runners, we search for a time at which all of them are away from the *starting point* by a distance of $(\frac{1}{n+1})$units. We can get that this distance is less than $(\frac{1}{n+1})$ units and we have the trivial lower bound of $\frac{1}{2n}$; for a proof of this lower bound, see [8], proposition 1.2. This lower bound is improved to $\frac{1}{2n} + \frac{c\log n}{n^2(\log\log n)^2}$ where c>0 is some absolute constant [8].

Also, we can assume the speed of all runners to be integers. This reduction, of the speeds from real to integers is done, for example in [4], section 4. In [7], it is proved that the lonely runner conjecture is true for two or more runners if we can assume that the speed of $(i+1)^{th}$ runner is more than double the speed of the i^{th} runner for each i, arranged in increasing order. We conclude by reproducing the proof of this result from [7].

Theorem1.

Let $S = \{s_1, s_2, \ldots, s_n\}$ *where* $n \geq 2$, *and* $(\frac{s_{i+1}}{s_i})\left(\frac{n-1}{n+1}\right) \geq 2$ *for each* $i = 1, 2, \ldots, n-1$. *Then there exists a real number x such that*

$$\| s_i x \| \geq \frac{1}{n+1} \qquad \textit{for each } i = 1, 2, \ldots, n.$$

Proof:

Consider an interval J = [a,b] = ($\frac{1}{s_{1(n+1)}}$, $\frac{n}{s_{1(n+1)}}$) Clearly, for $x \in J$, we have $\| s_1 x \| \geq \frac{1}{n+1}$ and the difference of the interval a-b = $\frac{n-1}{s_{1(n+1)}}$.

Let us denote the interval J by J_1.

Now construct the intervals $J_2, I_3, \ldots J_n$ with the following properties:

(a) $J_1 \supset J_2 \supset J_3 \supset \ldots \supset J_n$

(b) For $J_i = [a_i, b_i]$, $a_i - b_i = \frac{n-1}{s_{i(n+1)}}$

(c) For each $x \in J_i$, $\| s_i x \| \geq \frac{1}{n+1}$

Clearly, J_1 satisfies (b) and (c). Using induction, define the i^{th} interval $J_i = [a_i, b_i]$. We have

$$S_i b_{i-1} - s_i a_{i-1} = \left(\frac{s_i}{s_{i-1}}\right)\left(\frac{n-1}{n+1}\right) \geq 2.$$

Therefore, there exists an integer ℓ(i) such that

$$S_i a_{i-1} \leq \ell(i) < \ell(i) + 1 \leq s_i b_{i-1} \Rightarrow a_{i-1} \leq \frac{\ell(i)}{si} < \frac{\ell(i)+1}{si} \leq b_{i-1}$$

Define, $J_i = [a_i, b_i] = \left(\frac{\ell(i) + \frac{1}{n+1}}{s_i} , \frac{\ell(i) + \frac{n}{n+1}}{s_i} \right)$

The interval J_i satisfies all (a), (b) and (c). Since the intersection of the intervals $J_1, J_2, I_3, \ldots J_n$ is nonempty. Hence the proof. We conclude that the n runners having their speeds $s_1, s_2, \ldots s_n$ with $\left(\frac{s_{i+1}}{i}\right)\left(\frac{n-1}{n+1}\right) \geq 2$ satisfy the conjecture.

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